

# From graphs to free products

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## Abstract

We investigate a construction which associates a finite von Neumann algebra  $M(\Gamma, \mu)$  to a finite weighted graph  $(\Gamma, \mu)$ . Pleasantly, but not surprisingly, the von Neumann algebra associated to a ‘flower with  $n$  petals’ is the group von Neumann algebra of the free group on  $n$  generators. In general, the algebra  $M(\Gamma, \mu)$  is a free product, with amalgamation over a finite-dimensional abelian subalgebra corresponding to the vertex set, of algebras associated to subgraphs ‘with one edge’ (or actually a pair of dual edges). This also yields ‘natural’ examples of (i) a Fock-type model of an operator with a free Poisson distribution; and (ii)  $\mathbb{C} \oplus \mathbb{C}$ -valued circular and semi-circular operators.

## 1 Preliminaries

There has been a serendipitous convergence of investigations being carried out independently by us on the one hand, and by Guionnet, Jones and Shlyakhtenko on the other - see [GJS1], [KS1], [KS2], [GJS2]. As it has turned out, we have been providing independent proofs, from slightly different viewpoints, of the same facts. Both the papers [KS2] and [GJS2], establish that a certain von Neumann algebra associated to a graph is a free product with amalgamation of a family of von Neumann algebras corresponding to simpler graphs. The amalgamated product involved subgraphs indexed by vertices in [KS2], while the subgraphs are indexed by edges in [GJS2]. This paper was motivated by trying to understand how the proof of our result in [KS2] was also drastically simplified by considering edges rather than vertices. And, this third episode in our series seems to have the following points in its favour:

- It does make certain cumulant computations and consequent free independence assertions much more transparent.
- It brings to light a quite simple ‘Fock-type model’ of free Poisson variables.
- By allowing non-bipartite graphs, we get the aesthetically pleasing fact mentioned in the abstract regarding the ‘flower on  $n$  petals’.

We investigate, in a little more detail, the construction in [KS2] which associated a von Neumann probability space to a weighted graph. We begin by recalling the set-up:

By a weighted graph we mean a tuple  $\Gamma = (V, E, \mu)$ , where:

- $V$  is a (finite) set of vertices;
- $E$  is a (finite) set of edges, equipped with ‘source’ and ‘range’ maps  $s, r : E \rightarrow V$  and ‘(orientation) reversal’ involution map  $E \ni e \mapsto \tilde{e} \in E$  with  $(s(e), r(e)) = (r(\tilde{e}), s(\tilde{e}))$ ; and
- $\mu : V \rightarrow (0, \infty)$  is a ‘weight or spin function’ so normalised that  $\sum_{u \in V} \mu^2(u) = 1$

We let  $\mathcal{P}_n = \mathcal{P}_n(\Gamma)$  denote the set of paths of length  $n$  in  $\Gamma$  and let  $P_n(\Gamma)$  denote the vector space with basis  $\{[\xi] : \xi \in \mathcal{P}_n(\Gamma)\}$ . We think of  $\xi = \xi_1 \xi_2 \cdots \xi_n$  as the ‘concatenation product’ where  $\xi_i$  denotes the  $i$ -th edge of  $\xi$ . We write  $F(\Gamma) = \oplus_{n \geq 0} P_n(\Gamma)$  for the indicated direct sum, and equip it with the following slightly complicated multiplication: if  $\xi \in \mathcal{P}_m(\Gamma), \eta \in \mathcal{P}_n(\Gamma)$ , then  $[\xi] \# [\eta] = \sum_{k=0}^{\min(m,n)} \zeta_k$ , where  $\zeta_k \in \mathcal{P}_{m+n-2k}$  is defined by

$$\zeta_k = \begin{cases} \frac{\mu(v_m^\xi)}{\mu(v_{m-k}^\xi)} [\xi_1 \xi_2 \cdots \xi_{m-k} \eta_{k+1} \eta_{k+2} \cdots \eta_n] & \text{if } \xi_{m-j+1} = \tilde{\eta}_j \forall 1 \leq j \leq k \\ 0 & \text{otherwise} \end{cases}$$

Here, and elsewhere, we adopt the convention that if  $\xi \in \mathcal{P}_n$ , then  $\xi = \xi_1 \xi_2 \cdots \xi_n$  denotes concatenation product, with  $\xi_i \in E$  and we write  $s(\xi_i) = v_{i-1}^\xi$  (so also  $r(\xi_i) = s(\xi_{i+1}) = v_i^\xi$ ).

In particular, notice that  $\mathcal{P}_0(\Gamma) = \{v : v \in V\}$ , and that if  $v = s(\xi), w = r(\xi)$  for some  $\xi \in \mathcal{P}_n$ , and if  $u_1, u_2 \in V$ , then  $[u_1][\xi][u_2] = \delta_{u_1, v} \delta_{u_2, w} [\xi]$ ; and less trivially, if  $\xi \in \mathcal{P}_1$  and  $\eta \in \mathcal{P}_m, m \geq 1$ , then

$$[\xi] \# [\eta] = \begin{cases} 0 & \text{if } r(\xi) \neq s(\eta) \\ [\xi \eta_1 \cdots \eta_m] & \text{if } r(\xi) = s(\eta) \text{ but } \xi \neq \tilde{\eta}_1 \\ [\xi \eta_1 \cdots \eta_m] + \frac{\mu(r(\xi))}{\mu(s(\xi))} [\eta_2 \cdots \eta_m] & \text{if } \xi = \tilde{\eta}_1 \end{cases}$$

We define  $\phi : F(\Gamma) \rightarrow P_0$  by requiring that if  $\xi \in P_n$ , then

$$\phi([\xi]) = \begin{cases} 0 & \text{if } n > 0 \\ [\xi] & \text{if } n = 0 \end{cases}$$

and finally define

$$\tau = \mu^2 \circ \phi$$

where we simply write  $\mu^2$  for the linear extension to  $P_0(\Gamma)$  which agrees with  $\mu^2$  on the basis  $\mathcal{P}_0(\Gamma)$ .

It was shown in [KS]<sup>1</sup> that  $(F(\Gamma), \tau)$  is a tracial non-commutative \*-probability space, with  $e^* = \tilde{e}$ , that the mapping  $y \mapsto xy$  extends to a \*-algebra representation  $F(\Gamma) \rightarrow \mathcal{L}(L^2(F(\Gamma)), \tau)$  and that  $M(\Gamma, \mu) = \lambda(F\Gamma)'' \subset \mathcal{L}(L^2(F(\Gamma)), \tau)$  is in standard form. Before proceeding further, it is worth noting that for  $\xi, \eta \in \cup_n \mathcal{P}_n(\Gamma)$ , we have

$$\tau([\xi] \# [\eta]^*) = \delta_{\xi, \eta} \mu(r(\xi)) \mu(s(\xi)) ,$$

and hence, if we write  $\{\xi\} = (\mu(s(\xi)) \mu(r(\xi)))^{-\frac{1}{2}} [\xi]$ , then  $\{\{\xi\} : \xi \in \cup_{n \geq 0} \mathcal{P}_n(\Gamma)\}$  is an orthonormal basis for  $\mathcal{H}(\Gamma) = L^2(F(\Gamma), \tau)$ .

## 2 The building blocks

Our interest here is the examination of just how  $M(\Gamma, \mu)$  depends on  $(\Gamma, \mu)$ . We begin by spelling out some simple examples, which will turn out to be building blocks for the general case.

EXAMPLE 2.1. 1. Suppose  $|V| = |E| = 1$ , say  $V = \{v\}$  and  $E = \{e\}$ . Then we must have  $e = \tilde{e}$ ,  $s(e) = r(e) = v$ ,  $\mu(v) = 1$ ,  $\mathcal{P}_n = \{e^n\}$  and  $\{\xi(n) = \{e^n\} : n \geq 0\}$  (where  $\{e^0\} = \{v\}$ ) is an orthonormal basis for  $\mathcal{H}(\Gamma)$ ; and the definitions show that  $x = \lambda(e)$  satisfies  $x\xi_n = \xi(n+1) + \xi(n-1)$ . Thus  $x$  is a semi-circular element and  $M(\Gamma) = \{x\}'' \cong L\mathbb{Z}$ .

2. Suppose  $|V| = 1$ ,  $|E| = 2$ , say  $V = \{v\}$  and  $E = \{e_1, e_2\}$  suppose  $e_2 = \tilde{e}_1$ . Then we must have  $s(e_j) = r(e_j) = v$ ,  $\mu(v) = 1$ . Further  $\{\{e_1\}, \{e_2\}\}$  is an orthonormal basis for  $\mathcal{H}_2 = P_1(\Gamma)$ , and  $P_n(\Gamma)$  is isomorphic to  $\otimes^n \mathcal{H}_2$ . Thus  $\mathcal{H}(\Gamma)$  may be identified with the full Fock space  $\mathcal{F}(\mathcal{H}_2)$  and the definitions show that  $x_1 = \lambda(e_1)$  may be identified as  $x_1 = l_1 + l_2^*$ , where the  $l_j$  denote the standard creation operators. It follows that  $x_1$  is a circular element and  $M(\Gamma) = \{x_1\}'' \cong LF_2$ .

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<sup>1</sup>Actually, [KS] treated only the case of bipartite graphs, and sometimes restricted attention to the case of the Perron-Frobenius weighting; but for the the proof of statements made in this paragraph, none of those restrictions is necessary.

3. Suppose  $|V| = 2, |E| = 2$ , say  $V = \{v, w\}$  and  $E = \{e, \tilde{e}\}$  and suppose  $s(e) = v, r(e) = w$  and  $\mu(w) \leq \mu(v)$ . Write  $\rho = \frac{\mu(v)}{\mu(w)} (\geq 1)$ . If we let  $p_v = \lambda([v]), p_w = \lambda([w])$ , it follows that  $\mathcal{H}_v = \text{ran } p_v$  (resp.,  $\mathcal{H}_w = \text{ran } p_w$ ) has an orthonormal basis given by  $\{\{\eta(n)\} : n \geq 0\}$  (resp.,  $\{\{\xi(n)\} : n \geq 0\}$ ) where  $\eta(n) \in \mathcal{P}_n$  (resp.,  $\xi(n) \in \mathcal{P}_n$ ) and  $\eta(n)_k = e$  or  $\tilde{e}$  (resp.,  $\xi(n)_k = \tilde{e}$  or  $e$  according as  $k$  is odd or even).

Writing  $x = \lambda(e)$ , we see that with respect to the decomposition  $\mathcal{H}(\Gamma) = \mathcal{H}_v \oplus \mathcal{H}_w$ , the operator  $x$  has a matrix decomposition of the form

$$x = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$$

where  $t \in \mathcal{L}(\mathcal{H}_w, \mathcal{H}_v)$  is seen to be given by

$$\begin{aligned} t[\xi(n)] &= x[\xi(n)] \\ &= [e] \# [\tilde{e}e\tilde{e}e \cdots (n \text{ terms})] \\ &= [\eta(n+1)] + \rho^{-1}[\eta(n-1)] ; \end{aligned}$$

and hence,

$$\begin{aligned} t\{\xi(n)\} &= (\mu(s(\xi(n)))\mu(r(\xi(n))))^{-\frac{1}{2}} t[\xi(n)] \\ &= (\mu(w)\mu(r(\xi(n))))^{-\frac{1}{2}} ([\eta(n+1)] + \rho^{-1}[\eta(n-1)]) \\ &= (\rho^{-1}\mu(v)\mu(r(\eta(n \pm 1))))^{-\frac{1}{2}} ([\eta(n+1)] + \rho^{-1}[\eta(n-1)]) \\ &= \rho^{\frac{1}{2}}\{\eta(n+1)\} + \rho^{-\frac{1}{2}}\{\eta(n-1)\} \end{aligned}$$

It is a fact - see Proposition 2.2 - that  $t^*t$  has absolutely continuous spectrum. This fact has two consequences:

- (i) if  $t = u|t|$  is the polar decomposition of  $t$ , then  $u$  maps  $\mathcal{H}_w$  isometrically onto the subspace  $\mathcal{M} = \overline{\text{ran } t}$  of  $\mathcal{H}_v$ , and if  $z$  is the projection onto  $\mathcal{H}_v \ominus \mathcal{M}$  then  $\tau(z) = \mu^2(v) - \mu^2(w)$ ; and
- (ii)  $W^*(|t|) \cong L\mathbb{Z}$ .

Since  $p_v + p_w = 1$  and  $z \leq p_v$ , the definitions are seen to show that  $M(\Gamma, \mu)$  is isomorphic to  $\mathbb{C} \oplus M_2(L\mathbb{Z})$  via the unique isomorphism which maps  $p_v, p_w, z, u$  and  $|t|$ , respectively, to  $(1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}), (0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}), (1, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}), (0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$ , and  $(1, \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix})$  for some positive  $a$  with absolutely continuous spectrum which generates  $L\mathbb{Z}$  as a von Neumann algebra. (This

must be compared with Lemma 17 of [GJS2], bearing in mind that their  $\mu$  is our  $\mu^2$ .)

PROPOSITION 2.2. *Let  $\ell^2(\mathbb{N})$  have its standard orthonormal basis  $\{\delta_n : n \in \mathbb{N}\}$ . (For us,  $\mathbb{N} = \{0, 1, 2, \dots\}$ .) Let  $\ell\delta_n = \delta_{n+1}$  denote the creation operator (or unilateral shift), with  $\ell^*\delta_n = \delta_{n-1}$  (where  $\delta_{-1} = 0$ ). Let  $\rho > 1$  and  $t = \rho^{\frac{1}{2}}\ell + \rho^{-\frac{1}{2}}\ell^*$ . Then,*

1.  $t^*t$  leaves the subspace  $\ell^2(2\mathbb{N})$  invariant;
2.  $\delta_0$  is a cyclic vector for the restriction to  $\ell^2(2\mathbb{N})$  of  $t^*t$ , call it  $a_\rho$ ; and
3. the (scalar) spectral measure of  $a_\rho$  associated to  $\delta_0$  is absolutely continuous with respect to Lebesgue measure; in fact  $a_\rho$  has a free Poisson distribution.

*Proof.* A little algebra shows that

$$\begin{aligned} t^*t &= (\rho^{\frac{1}{2}}\ell^* + \rho^{-\frac{1}{2}}\ell)(\rho^{\frac{1}{2}}\ell + \rho^{-\frac{1}{2}}\ell^*) \\ &= \ell^2 + \ell^{*2} + (\rho + \rho^{-1}) - \rho^{-1}p_0, \end{aligned}$$

where  $p_0$  is the rank one projection onto  $\mathbb{C}\delta_0$ . It is seen that this operator leaves both subspaces  $\ell^2(2\mathbb{N})$  and  $\ell^2(2\mathbb{N}+1)$  invariant, with its restrictions to these subspaces being unitarily equivalent to  $\ell + \ell^* + (\rho + \rho^{-1}) - \rho^{-1}p_0$  and  $\ell + \ell^*$  respectively. Since the spectral type does not change under scalar translation, we may assume without loss of generality that  $a_\rho = \ell + \ell^* - \rho^{-1}p_0$  and establish that  $a_0$  has absolutely continuous scalar spectral measure corresponding to  $\delta_0$ .

Write  $a_0 = \ell + \ell^*$  so that  $a_\rho = a_0 - \rho^{-1}p_0$ . Let the scalar spectral measures of  $a_0$  and  $a_\rho$  be denoted by  $\mu$  and  $\mu_\rho$  respectively, and consider their Cauchy transforms given by

$$F_\lambda(z) = \langle (a_\lambda - z)^{-1}\delta_0, \delta_0 \rangle = \int_{\mathbb{R}} \frac{d\mu_\lambda(x)}{x - z}$$

for  $\lambda \in \{0, \rho\}$  and  $z \in \mathbb{C}^+ = \{\zeta \in \mathbb{C} : \text{Im}(\zeta) > 0\}$ .

It follows from the resolvent equation that

$$\begin{aligned} F_\rho(z) &= \langle (a_\rho - z)^{-1}\delta_0, \delta_0 \rangle \\ &= \langle (a_0 - z)^{-1}\delta_0, \delta_0 \rangle + \langle (a_\rho - z)^{-1}\rho^{-1}p_0(a_\lambda - z)^{-1}\delta_0, \delta_0 \rangle \\ &= F_0(z) + \rho^{-1}F_\rho(z)F_0(z); \end{aligned}$$

Hence

$$F_\rho(z) = \frac{F_0(z)}{1 - \rho^{-1}F_0(z)} = \frac{\rho F_0(z)}{\rho - F_0(z)} \quad (2.1)$$

It is seen from Lemma 2.21 of [NS] - after noting that the  $G$  of that Lemma is the negative of the  $F_0$  here - that  $F_0(z) = \frac{z - \sqrt{z^2 - 4}}{2}$  where  $\sqrt{z^2 - 4}$  is a branch of that square root such that  $\sqrt{z^2 - 4} = \sqrt{z + 2}\sqrt{z - 2}$  where the two individual factors are respectively defined by using the branch-cuts  $\{\mp 2 - it : t \in (0, \infty)\}$ . (This choice ensures that  $\lim_{|z| \rightarrow \infty} F_0(z) = 0$ , which is clearly necessary.) It follows that  $F_0$ , which is holomorphic in  $\mathbb{C}^+$ , actually extends to a continuous function on  $\mathbb{C}^+ \cup \mathbb{R}$ , and that if we write  $f_0(a) = \lim_{b \downarrow 0} F_0(a + ib)$ , then we have

$$2f_0(t) = \begin{cases} -t + \sqrt{t^2 - 4} & \text{if } t \geq 2 \\ -t + i\sqrt{4 - t^2} & \text{if } t \in [-2, 2] \\ -t - \sqrt{t^2 - 4} & \text{if } t \leq -2 \end{cases} \quad (2.2)$$

It is easy to check that  $f_0$  is strictly increasing in  $(-\infty, -2)$ , as well as in  $(2, \infty)$ , has non-zero imaginary part in  $(-2, 2)$ , and satisfies  $f(\mathbb{R} \setminus (-2, 2)) = [-1, 0) \cup (0, 1]$ . Since  $\rho > 1$ , we may deduce that  $F_0(z) \neq \rho \ \forall z \in \mathbb{C}^+ \cup \mathbb{R}$ , and hence that also  $F_\rho$  extends to a continuous function on  $\mathbb{C}^+ \cup \mathbb{R}$  with equation (2.1) continuing to hold for all  $z \in \mathbb{C}^+ \cup \mathbb{R}$ . Writing  $f_\lambda(t) = F_\lambda(t + i0)$  for  $\lambda \in \{0, \rho\}$ , we find that

$$f_\rho(t) = \frac{\rho f_0(t)}{\rho - f_0(t)} = \frac{1}{f_0(t)^{-1} - \rho^{-1}} ,$$

and hence that

$$\begin{aligned} \operatorname{Im}(f_\rho(t)) &= -\frac{\operatorname{Im}(f_0(t)^{-1})}{|f_0(t)^{-1} - \rho^{-1}|^2} \\ &= \frac{\operatorname{Im}(f_0(t))}{|1 - f_0(t)\rho^{-1}|^2} \\ &= \rho^2 \frac{\operatorname{Im}(f_0(t))}{|f_0(t) - \rho|^2} \\ &= 1_{[-2, 2]}(t) \frac{\rho^2 \sqrt{4 - t^2}}{2|f_0(t) - \rho|^2} . \end{aligned}$$

Now, for  $t \in [-2, 2]$ , we see that

$$\begin{aligned} |f_0(t) - \rho|^2 &= \left| \frac{-t + i\sqrt{4 - t^2}}{2} - \rho \right|^2 \\ &= \frac{1}{4} ((t + 2\rho)^2 + 4 - t^2) \\ &= \rho^2 + \rho t + 1 . \end{aligned}$$

It follows from Stieltje's inversion formula that our  $a_\rho$  has absolutely continuous scalar spectral measure  $\mu_\rho$ , with density given

by

$$\begin{aligned} g_\rho(t) &= \frac{1}{\pi} \text{Im} f_\rho(t) \\ &= 1_{[-2,2]}(t) \frac{\rho^2 \sqrt{4-t^2}}{2\pi(\rho^2 + \rho t + 1)} . \end{aligned}$$

Hence the operator  $t^*t = a_\rho + (\rho + \rho^{-1})1$  has absolutely continuous scalar spectral measure, with density given by

$$\begin{aligned} g(t) &= g_\rho(t - (\rho + \rho^{-1})) \\ &= 1_{[(\rho+\rho^{-1})-2, (\rho+\rho^{-1})+2]}(t) \frac{\rho^2 \sqrt{4 - (t - (\rho + \rho^{-1}))^2}}{2\pi \rho^{-2}(\rho^2 + \rho(t - \rho - \rho^{-1}) + 1)} \\ &= 1_{[(\rho+\rho^{-1})-2, (\rho+\rho^{-1})+2]}(t) \frac{\rho^2 \sqrt{4 - (t - (\rho + \rho^{-1}))^2}}{2\pi \rho^{-1}t} \end{aligned}$$

If we write  $\lambda = \rho^2$  and  $\alpha = \rho^{-1}$ , we see that  $\alpha(1 + \lambda)$  and recognise the fact that not only does  $t^*t$  have absolutely continuous spectrum, but - by comparing with equation (12.15) of [NS] - even that it actually has a free Poisson distribution, with rate  $\rho^2$  and jump size  $\rho^{-1}$ . However, we actually discovered this fact about  $t^*t$  having a free Poisson distribution with the stated  $\lambda$  and  $\alpha$  by a cute cumulant computation which we present in the final section, both for giving a combinatorial rather than analytic proof of this Proposition, and because we came across that proof first.  $\square$

### 3 Some free cumulants

Before proceeding with the further study of a general  $(\Gamma, \mu)$ , we will need an alternative description of  $M(\Gamma, \tau)$ .

Let  $Gr(\Gamma) = \oplus_{n \geq 0} P_n(\Gamma)$  be equipped with a  $*$ -algebra structure wherein  $[\xi] \circ [\eta] = [\xi\eta]$  and  $[\xi]^* = [\tilde{\xi}] = [\tilde{\xi}_n \cdots \tilde{\xi}_1]$  for  $\xi \in P_n, \eta \in P_m$ . It turns out - see [KS]<sup>2</sup> - that  $Gr(\Gamma)$  and  $F(\Gamma)$  are isomorphic as  $*$ -algebras. While the multiplication is simpler in  $Gr(\Gamma)$ , the trace  $\tau$  on  $F(\Gamma)$  turns out, when transported by the above isomorphism, to be given by a slightly more complicated formula. (It is what has been called *the Voiculescu trace* by Jones et al.) We shall write  $tr$  for this transported trace on  $Gr(\Gamma)$ , and  $\phi$  for the  $tr$ -preserving conditional expectation of  $M(\Gamma, \mu)(= \lambda(Gr(\Gamma)))''$  onto  $P_0(\Gamma)$ . We shall use the

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<sup>2</sup>The remark made in an earlier footnote, concerning assumptions regarding bipartiteness of  $\Gamma$ , applies here as well.

same letter  $\phi$  to denote restrictions to subalgebras which contain  $P_0(\Gamma)$ .

We wish to regard  $(Gr(\Gamma), \phi)$  as an operator-valued non-commutative probability space over  $P_0(\Gamma)$ , our first order of business being the determination of the  $P_0(\Gamma)$ -valued mixed cumulants in  $Gr(\Gamma)$ .

**PROPOSITION 3.1.** *The  $P_0(\Gamma)$ -valued mixed cumulants in  $Gr(\Gamma)$  are given thus:*

*$\kappa_n([e_1], [e_2], \dots, [e_n]) = 0$  unless  $n = 2$  and  $e_2 = \tilde{e}_1$ ; and if  $e_2 = \tilde{e}_1$  with  $s(e_1) = v, r(e_1) = w$ , then  $\kappa_2([e_1], [\tilde{e}_1]) = \frac{\mu(w)}{\mu(v)}[v]$ .*

*Proof.* The proof depends on the ‘moment-cumulant’ relations which guarantee that in order to prove this proposition, it will suffice to establish the following, which is what we shall do:

(a) Define  $\kappa_n : (Gr(\Gamma))^n \rightarrow P_0(\Gamma)$  to be the unique multilinear map which is defined when the arguments are tuples of paths as asserted in the proposition; note that (i) it is ‘balanced’ over  $P_0(\Gamma)$  in the sense that  $\kappa_n(x_1, \dots, x_{i-1}b, x_i, \dots, x_n) = \kappa_n(x_1, \dots, x_{i-1}, bx_i, \dots, x_n)$  for all  $x_j \in Gr(\Gamma), b \in P_0(\Gamma)$  and  $1 < i \leq n$ , and (ii) is  $P_0(\Gamma)$ -bilinear meaning  $\kappa_n(bx_1, x_2, \dots, x_{n-1}, x_nb') = b\kappa_n(x_1, x_2, \dots, x_{n-1}, x_n)b'$  for all  $x_j \in Gr(\Gamma), b, b' \in P_0(\Gamma)$ ;

(b) define the ‘multiplicative extensions’  $\kappa_\pi : (Gr(\Gamma))^n \rightarrow P_0(\Gamma)$  for  $\pi \in NC(n)$  by requiring, inductively, that if  $[k, l]$  is an interval constituting a class of  $\pi$ , and if we write  $\sigma$  for the element of  $NC(n - l + k - 1)$  given by the restriction of  $\pi$  to  $\{1, \dots, k-1, l+1, \dots, n\}$ , so that ‘ $\pi = \sigma \vee 1_{[k, l]}$ ’ then

$$\begin{aligned} \kappa_\pi(x_1, \dots, x_n) &= \kappa_\sigma(x_1, \dots, x_{k-1}\kappa_{l-k+1}(x_k, \dots, x_l), x_{l+1}, \dots, x_n) \\ &= \kappa_\sigma(x_1, \dots, x_{k-1}, \kappa_{l-k+1}(x_k, \dots, x_l)x_{l+1}, \dots, x_n); \end{aligned}$$

(c) and verify that for any  $e_1, \dots, e_n \in \mathcal{P}_1(\Gamma)$ ,

$$\phi([e_1] \cdots [e_n]) = \sum_{\pi \in NC(n)} \kappa_\pi([e_1], [e_2], \dots, [e_n]). \quad (3.3)$$

For this verification, we first assert that if  $e_1, e_2, \dots, e_n \in E$  and  $\pi \in NC(n)$ , the quantity  $\kappa_\pi([e_1], [e_2], \dots, [e_n])$  (yielded by the unique ‘multiplicative extension’ of the  $\kappa_n$ ’s as in (b) above) can be non-zero only if

- (i)  $e_1 e_2 \cdots e_n$  is a meaningfully defined loop based at  $s(e_1)$ , meaning  $f(e_i) = s(e_{i+1})$  for  $1 \leq i \leq n$ , with  $e_{n+1}$  being interpreted as  $e_1$ ;
- (ii)  $\pi \in NC_2(n)$  is a pair partition of  $n$  (and in particular  $n$  is even), such that  $\{i, j\} \in \pi \Rightarrow e_j = \tilde{e}_i$ ;



and if that is the case, then,

$$\kappa_\pi([e_1], [e_2], \dots, [e_n]) = \left( \prod_{\substack{\{i,j\} \in \pi \\ i < j}} \frac{\mu(r(e_i))}{\mu(r(e_j))} \right) [s(e_1)] . \quad (3.4)$$

We prove this assertion by induction on  $n$ . This is trivial for  $n = 1$  since  $\kappa_1 \equiv 0$ . By the inductive definition of the multiplicative extension, it is clear that if  $\kappa_\pi([e_1], [e_2], \dots, [e_n])$  is to be non-zero,  $\pi$  must contain an interval class of the form  $\{k, k+1\}$  such that  $e_{k+1} = \tilde{e}_k$ ; if  $\sigma$  denotes  $\pi|_{\{1,2,\dots,k-1,k+2,\dots,n\}}$  we must have

$$\begin{aligned} \kappa_\pi([e_1], \dots, [e_n]) &= \frac{\mu(r(e_k))}{\mu(r(e_{k+1}))} \kappa_\sigma([e_1], \dots, [e_{k-1}], [s(e_k)], [e_{k+2}], \dots, [e_n]) \\ &= \frac{\mu(r(e_k))}{\mu(r(e_{k+1}))} \kappa_\sigma([e_1], \dots, [e_{k-1}], [s(e_k)][e_{k+2}], \dots, [e_n]) \\ &= \frac{\mu(r(e_k))}{\mu(r(e_{k+1}))} \kappa_\sigma([e_1], \dots, [e_{k-1}], [r(e_{k+1})], [e_{k+2}], \dots, [e_n]) ; \end{aligned}$$

and for this to be non-zero, we must have  $r(e_{k-1}) = s(e_k) = r(e_{k+1}) = s(e_{k+2})$ , in which case we would have

$$\kappa_\pi([e_1], \dots, [e_n]) = \frac{\mu(r(e_k))}{\mu(r(e_{k+1}))} \kappa_\sigma([e_1], \dots, [e_{k-1}], [e_{k+2}], \dots, [e_n]) ,$$

and the requirement that  $\kappa_\sigma([e_1], \dots, [e_{k-1}], [e_{k+2}], \dots, [e_n])$  be non-zero, along with the induction hypothesis, finally completes the proof of the assertion.

Now, in order to verify equation 3.3, it suffices to check that for any  $v \in V$ , we have

$$tr([e_1][e_2] \cdots [e_n][v]) = \sum_{\pi \in NC(n)} tr(\kappa_\pi([e_1], [e_2], \dots, [e_n])[v]). \quad (3.5)$$

First observe that both sides of equation 3.5 vanish unless  $e_1 \cdots e_n$  is a meaningfully defined path with both source and range equal to  $v$  (since  $tr$  is a trace and  $[v]$  is idempotent). In view of our description above of the multiplicative extension  $\kappa_\pi$ , we need, thus, to verify that for such a loop, we have

$$tr([e_1] \cdots [e_n]) = \sum_{\pi \in NC_2(n)} \left( \prod_{\substack{\{i,j\} \in \pi \\ i < j}} \delta_{e_j, \tilde{e}_i} \frac{\mu(r(e_i))}{\mu(r(e_j))} \right) \mu^2(s(e_1)),$$

but that is indeed the case (see equation (3) and the proof of Proposition 5 in [KS1]).  $\square$

In order to derive the true import of Proposition 3.1, we should first introduce some notation:

For each dual pair  $e, \tilde{e}$  of edges - with, say,  $s(e) = v, r(e) = w$  - we shall write  $\Gamma_e = (V_e, E_e, \mu_e)$  where  $V_e = V, \mu_e = \mu$  and  $E_e = \{e, \tilde{e}\}$  (with source, range and reversal in  $E_e$  as in  $E$ ). If  $e = \tilde{e}$ , the above definitions are to be suitably interpreted. Now for ‘the true import of Proposition 3.1’:

**COROLLARY 3.2.** *With the foregoing notation, we have:*

$$Gr(\Gamma, \mu) = *_{P_0(\Gamma)} \{Gr(\Gamma_e, \mu_e) : \{e, \tilde{e}\} \subset E\}$$

and hence, also

$$M(\Gamma, \mu) = *_{P_0(\Gamma)} \{M(\Gamma_e, \mu_e) : \{e, \tilde{e}\} \subset E\} .$$

*Proof.* Proposition 3.3.3 of [S1] shows that if  $A \xrightarrow{\phi} B$  is a ‘non-commutative probability space over  $B$ ’, if  $\{A_i : i \in I\}$  is a family of subalgebras of  $A$  containing  $B$ , such that  $\{A_i : i \in I\}$  generates  $A$ , and if  $G_i$  is a set of generators of the algebra  $A_i$ , then  $A$  is the free product with amalgamation over  $B$  of  $\{A_i : i \in I\}$  if and only if the mixed  $B$ -valued cumulants  $\kappa_n(x_1, \dots, x_n)$  vanish whenever  $x_1, \dots, x_n \in \cup_i G_i$ , unless all the  $x_i$  belong to the same  $G_k$  for some  $k$ . The desired assertion then follows from Proposition 3.1.  $\square$

The following assertion, advertised in the abstract, is an immediate consequence of Corollary 3.2 and Examples 2.1 (1) and (2).

**COROLLARY 3.3.** *If  $\Gamma_n$  denotes the ‘flower with  $n$  petals’ (thus  $|V| = 1, |E| = n$ ), then  $M(\Gamma) \cong L\mathbb{F}_n$ , independent of the reversal map on  $E$ .*

**REMARK 3.4.** *We may deduce from Proposition 3.1 that the  $x = \lambda(e)$  of Example 2.1 (3) is a  $P_0(\Gamma)$ -valued circular operator, in the sense of [Dyk] (see Definition 4.1), with covariance  $(\alpha, \beta)$  where  $\alpha(b) = \phi(x^*bx)$  and  $\beta(b) = \phi(xbx^*)$  for all  $b \in \mathcal{P}_0$  are the completely positive self-maps of  $P_0(\Gamma) (= \mathbb{C}p_v \oplus \mathbb{C}p_w)$  induced by the matrices*

$$\alpha = \begin{bmatrix} 0 & \rho^{-1} \\ 0 & 0 \end{bmatrix} \text{ and } \beta = \begin{bmatrix} 0 & 0 \\ \rho & 0 \end{bmatrix} .$$

If  $s = x + x^*$ , it follows then that  $s$  is a  $P_0(\Gamma)$ -valued semi-circular element (since  $\kappa_n(sb_1, sb_2, \dots, sb_{n-1}, s) = 0$  unless  $n = 2$  and  $\kappa_2(sb, s) = \eta(b)$  where  $\eta$  is the (completely) positive self-map of  $\mathbb{C} \oplus \mathbb{C}$  induced by the matrix

$$\eta = \begin{bmatrix} 0 & \rho^{-1} \\ \rho & 0 \end{bmatrix}.$$

## 4 Narayana numbers

Recall the Narayana numbers  $N(n, k)$  defined for all  $n, k \in \mathbb{N}$  with  $1 \leq k \leq n$  by

$$N(n, k) = |\{\pi \in NC(n) : |\pi| = k\}|.$$

Define the associated polynomials  $N_n$  by

$$N_n(T) = \sum_{k=1}^n N(n, k) T^k.$$

Recall also that a random variable in a non-commutative probability space  $(A, \tau)$  is said to be free Poisson with rate  $\lambda$  and jump size  $\alpha$  if its free cumulants are given by  $\kappa_n = \lambda \alpha^n$  for all  $n \in \mathbb{N}$ . An easy application of the moment-cumulant relations shows that an equivalent condition for a random variable to be free Poisson with rate  $\lambda$  and jump size  $\alpha$  is that its moments are given by  $\mu_n = \alpha^n N_n(\lambda)$  for all  $n \in \mathbb{N}$ .

We now illustrate an application of this characterisation of a free Poisson variable in the situation of §2, Example 2.1 (3). There,  $x = \lambda(e)$  has a matrix decomposition involving  $t \in \mathcal{L}(\mathcal{H}_w, \mathcal{H}_v)$  where  $t^*t$  was shown to have a free Poisson distribution. We will verify below by a cumulant computation that  $t^*t$  is free Poisson with rate  $\rho^2$  and jump size  $\rho^{-1}$  in the non-commutative probability space  $p_w M(\Gamma, \mu) p_w$ .

Begin by observing that  $x^*x$  has a non-zero entry only in the  $w$ -corner and that this entry is  $t^*t$ . Thus the trace in  $M(\Gamma, \mu)$  of  $(x^*x)^n$  is  $\mu^2(w)$  times the trace - call it  $tr_w$  - in  $p_w M(\Gamma, \mu) p_w$  of  $(t^*t)^n$ . We now compute  $tr((x^*x)^n) = tr([e]^*[e]^n)$ .

First apply the moment-cumulant relations and Proposition 3.1 to conclude that

$$\phi([e]^*[e]^n) = \sum_{\pi \in NC(2n)} \kappa_\pi([e]^*, [e], \dots, [e]^*, [e]).$$

While this sum ranges over all  $\pi \in NC(2n)$ , Proposition 3.1 enables us to conclude that unless  $\pi$  is a non-crossing pair partition, its contribution vanishes. Thus we have:

$$\phi([e]^*[e])^n = \sum_{\pi \in NC_2(2n)} \kappa_\pi([e]^*, [e], \dots, [e]^*, [e]).$$

Now we use the well-known bijection between non-crossing pair partitions (or equivalently, Temperley-Lieb diagrams) on  $2n$  points and all non-crossing partitions on  $n$  points. We will denote this bijection as  $\pi \in NC_2(2n) \leftrightarrow \tilde{\pi} \in NC(n)$ . This is illustrated by example in Figure 4 for  $\pi = \{\{1, 8\}, \{2, 5\}, \{3, 4\}, \{6, 7\}, \{9, 12\}, \{10, 11\}\}$  and may be summarised by saying that the black regions of the Temperley-Lieb diagram for  $\pi \in NC_2(2n)$  correspond to the classes of  $\tilde{\pi} \in NC(n)$ . Note that in Figure 4 the numbers above refer to the

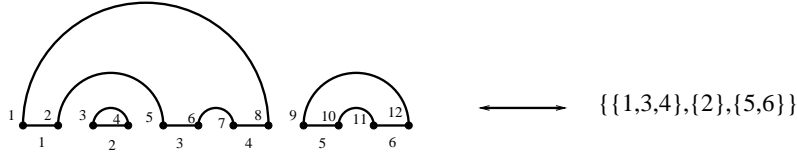


Figure 1:  $\pi \in NC_2(12) \leftrightarrow \tilde{\pi} \in NC(6)$

vertices while those below refer to the black segments.

It follows from Proposition 3.1 that for any  $\pi \in NC_2(2n)$ , the term  $\kappa_\pi([e]^*, [e], \dots, [e]^*, [e])$  is a scalar multiple of  $p_w$  where the scalar is given by a product of  $n$  terms each of which is  $\rho = \frac{\mu(v)}{\mu(w)}$  or  $\rho^{-1} = \frac{\mu(w)}{\mu(v)}$ . Classes of  $\pi$  for which the smaller element is odd give  $\rho$ , while those for which the smaller element is even give  $\rho^{-1}$ . Thus  $\kappa_\pi([e]^*, [e], \dots, [e]^*, [e])$  evaluates to  $\rho^{(|\pi|_{\text{odd}} - |\pi|_{\text{even}})} p_w = \rho^{(2|\pi|_{\text{odd}} - n)} p_w$ , where, of course,  $|\pi|_{\text{odd}}$  (resp.  $|\pi|_{\text{even}}$ ) denotes the number of classes of  $\pi$  whose smaller element is odd (resp. even).

Our main combinatorial observation is contained in the following simple lemma.

LEMMA 4.1. *For any  $\pi \in NC_2(2n)$ ,  $|\pi|_{\text{odd}} = |\tilde{\pi}|$ .*

*Proof.* We induce on  $n$  with the basis case  $n = 1$  having only one  $\pi$  with  $|\pi|_{\text{odd}} = |\tilde{\pi}| = 1$ . For larger  $n$ , consider a class of  $\pi$  of the form  $\{i, i+1\}$ , and remove it to get  $\rho \in NC_2(2n-2)$ . A moment's thought shows that if  $i$  is odd then  $|\pi|_{\text{odd}} = |\rho|_{\text{odd}} + 1 = |\tilde{\rho}| + 1 = |\tilde{\pi}|$ , while if  $i$  is even then  $|\pi|_{\text{odd}} = |\rho|_{\text{odd}} = |\tilde{\rho}| = |\tilde{\pi}|$ .  $\square$

Thus:

$$\begin{aligned}
\phi([e]^*[e])^n &= \sum_{\pi \in NC_2(2n)} \rho^{(2|\pi|_{\text{odd}}-n)} p_w \\
&= \sum_{\tilde{\pi} \in NC(n)} \rho^{(2|\tilde{\pi}|-n)} p_w \\
&= \sum_{k=1}^n \sum_{\{\tilde{\pi} \in NC(n): |\tilde{\pi}|=k\}} \rho^{2k-n} p_w \\
&= \sum_{k=1}^n N(n, k) \rho^{2k-n} p_w
\end{aligned}$$

Hence  $\text{tr}([e]^*[e])^n = \sum_{k=1}^n N(n, k) \rho^{2k-n} \mu^2(w)$  and thus  $\text{tr}_w((t^*t)^n) = \sum_{k=1}^n N(n, k) \rho^{2k-n}$ . Now the characterisation of free Poisson elements in terms of their moments shows that  $t^*t$  is free Poisson with rate  $\rho^2$  and jump size  $\rho^{-1}$ .

- REMARK 4.2. 1. Thus, for  $t = \rho^{\frac{1}{2}}\ell + \rho^{-\frac{1}{2}}\ell^*$ , we have shown that  $t^*t$  is a free Poisson element with rate  $\rho^2$  and jump size  $\rho^{-1}$ . By scaling by an appropriate constant, we can similarly obtain such simple Fock-type models of free Poisson elements with arbitrary jump size and rate.
2. Similar scaling, and the fact that  $e^{i\theta}\ell$  is unitarily equivalent to  $\ell$  (by a unitary operator which fixes  $\delta_0$ ) show that, in fact, if  $t = a\ell + b\ell^*$  for any  $a, b \in \mathbb{C}$ , then  $t^*t$  is a free Poisson element.

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